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Tall tales from de Sitter space II: field theory dualities

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Abstract: We consider the evolution of massive scalar fields in (asymptotically) de Sitter spacetimes of arbitrary dimension. Through the proposed dS/CFT correspondence, our analysis points to the existence of new nonlocal dualities for the euclidean conformal field theory. A massless conformally coupled scalar field provides an example where the analysis is easily explicitly extended to 'tall' background spacetimes.

Keywords: Duality in Gauge Field Theories, AdS-CFT and dS-CFT Correspondence, Models of Quantum Gravity.

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1. Introduction

Observations [1] suggest that our universe is proceeding towards a phase where its evolution is dominated by a small positive cosmological constant. This suggestion poses new challenges for string theory, which has seen much success in asymptotically flat spaces and in settings with an effective negative cosmological constant (e.g., Freund-Rubin compactifications [2] producing asymptotically anti-de Sitter spaces). The impressive success of the AdS/CFT correspondence [3], which has provided fairly concrete realizations of a holographic duality between quantum gravity and a field theory in one dimension lower without gravity, has prompted speculations that it may be possible to describe string theory or quantum gravity in asymptotically de Sitter spaces by a similar dS/CFT correspondence [4, 5]. Like the AdS case, the symmetries of de Sitter space suggest that the dual field theory is conformally invariant.

Of course, the nature of de Sitter space is quite different from its AdS counterpart. In particular the conformal boundaries, which one expects to play a central role in any dS/CFT correspondence, are hypersurfaces of euclidean signature. As a result, one expects the dual field theory to be a euclidean field theory. Further in de Sitter space, there are two such hypersurfaces: the future boundary, I^+, and the past boundary, I^-. Hence one must ask whether the proposed duality will involve a single field theory [4, 5] or two [6]. Unfortunately at present, the most striking difference from the AdS/CFT duality is the fact that we have no rigorous realizations of the dS/CFT duality — some progress has been made with pure three-dimensional de Sitter gravity [7].

However, the idea of a dS/CFT correspondence is a powerful and suggestive one that could have fundamental implications for the physics of our universe. A present difficulty is that rather little is known about the euclidean field theory which is to be dual to physics in the bulk. Our goal here is to explore further the requirements for a candidate dual field theory under the assumption that the bulk physics must reproduce standard background quantum field theory in the low energy limit. In particular then, up to possible quantum gravity violations, it should be possible to express the observables[1] localized near any Cauchy surface in terms of the observables localized near any other Cauchy surface. Of course, this is what one usually refers to as 'unitarity,' though even in standard quantum field theory it may not represent unitary evolution in the technical sense [10, 11]. Below, we will find that this benign assumption about the bulk physics has rather extraordinary consequences for the dual CFT.

Our primary tool for exploring these consequences is Einstein gravity with a positive cosmological constant coupled to an otherwise free scalar field. For simplicity, we consider quantum field theory for the scalar linearized about fixed de Sitter or asymptotically de Sitter backgrounds. However, it will be clear that, at least as far as the above property is concerned, linearized gravitons and even non-linear background quantum field theory must yield similar results. Before proceeding with a detailed discussion of the scalar field, let us briefly present our main observations.

Central to our investigations are the two conformal boundaries appearing in de Sitter space, and the claim [4, 5] that only a single dual field theory is needed. Recently, an alternate point of view has been advocated [6] in which the dual description of dS space involves an entangled state of two field theories associated with the two separate boundaries. We will comment on this idea in the discussion (section 4), but our present focus will be on the single CFT proposal where one does not have a 'separate' dual theory associated with I^+ and with I^-. Some intuition for this point of view can be found by combining the above assumption of bulk unitarity with the central postulate of the dS/CFT correspondence. That is, operators in the dual field theory may be associated with appropriate limits of bulk operators pushed to either I^- or I^+. So consider the full set of bulk operators localized near some arbitrary Cauchy surface. Now unitary time evolution in the bulk allows us to 'push' this complete set of operators to either I^- or I^+ and hence obtain corresponding sets of dual operators. Of course, since we began with a complete set of bulk operators, correlation functions for any bulk operators can then be computed in terms of those of the dual operators on either boundary. In particular, correlation functions in the dual theory associated with I^+ can be expressed in terms of correlation functions associated with I^- and vice versa.

[1]Here we use the term 'observables' in the technical sense of 'gauge invariant operators' without direct concern for a sense in which these observables might be measurable [8, 9].

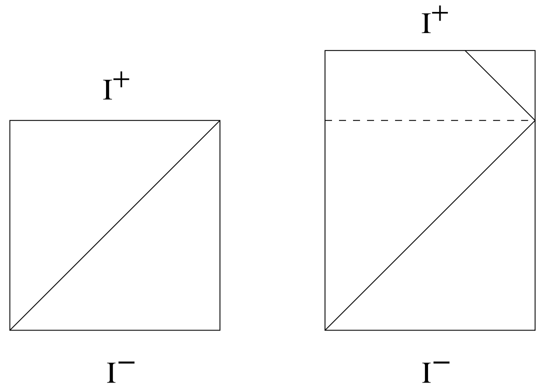

Figure 1: Conformal diagrams of *a*) de Sitter space and *b*) perturbed de Sitter space.

The original discussion of [4, 5] emphasized the causal connection between points on the two boundaries of dS space. In particular, a light cone emerging from a point on I^- expands into the space and reconverges at the antipodal point on the sphere at I^+. As a consequence, the singularity structure of certain boundary correlation functions is left invariant when, e.g., a local operator on I^- is replaced by a corresponding local operator at the antipodal point on I^+ [4], as will be reviewed in section 3. Not only does this observation suggest that there is a single dual field theory, but further that dual operators associated with the two boundaries are simply related by the antipodal map on the sphere. The latter would be surprising as one expects that in general the time evolution map connecting the corresponding bulk operators is nonlocal. In fact, as will be discussed below, this latter intuition is correct. The mapping between dual operators on $I^\pm$ is highly nonlocal, as is readily revealed by explicitly examining the (retarded) Green's functions in dS space. While the singularities in the Green's function propagate along the light cone, generically there is also nontrivial support within the light cone. For certain special cases, however, the time evolution map does produce a simply antipodal mapping between I^+ and I^-. Given the lack of guidance coming from a working example of the proposed duality, one might interpret this result as a hint towards the specific types of fields that would appear in a successful realization of the dS/CFT. Unfortunately, however, this selection rule based on locality of the mapping between boundaries does not seem to bear up in more interesting applications, as follows.

The discussion of the dS/CFT correspondence has also been extended beyond pure dS space to more general backgrounds with asymptotically dS regions. Again, in analogy to the AdS/CFT duality, such backgrounds might have an interpretation in terms of 'renormalization group flows' in the dual field theory [12]–[16]. Now it is a straightforward consequence of a theorem of Gao and Wald [17] that in such a generic (nonsingular) background, observers are able to view an entire Cauchy surface at a finite time. The corresponding conformal diagrams may be described as 'tall' (see figure 1 — see also [14] for details).

The nonlocality in the relation between dual operators on the two boundaries becomes manifest when we consider such 'tall' backgrounds. The above discussion, in which the bulk evolution map relates the CFT operators associated with I^+ and I^-, remains essentially unchanged. However, a key difference is that the causal connection between I^- and I^+ is now manifestly *not* local. In a tall spacetime, the light rays emerging from a point on I^- reconverge,[2] but this occurs at a finite time long before they reach I^+. After passing through the focal point, the rays diverge again to enclose a finite region on I^+. This observation precludes any intuition that the dual operators associated with the two boundaries could be related by a simple local map (e.g., the antipodal map) on the sphere. That is, following the arguments presented above, one is still lead to conclude that in these tall backgrounds pushing the bulk operators to I^- and I^+ must yield equivalent dual physics, but the map that implements this equivalence can no longer be local. Hence in the context of the dS/CFT duality, it seems that nonlocality will be an unavoidable aspect of the relation between field theory operators associated with two conformal boundaries. One can reproduce precisely the same boundary correlators or observables but through some nonlocal reorganization of the degrees of freedom within the dual field theory. It seems appropriate to refer to such relations as *nonlocal dualities* within the field theory.

The remainder of the paper is organized as follows: We begin with a brief review of the dS/CFT correspondence in section 2. This also allows us to emphasize certain subtleties that arise in this proposal, and it sets the stage for our main discussion. In section 3, we explore in more detail the consequences of the bulk evolution map for the relationship between the dual field theories associated with the two boundaries, I^- and I^+. Section 4 provides a short discussion of our results. Various useful technical details about scalar field theory in de Sitter space are presented in appendices. Appendix A provides a detailed analysis of the evolution of massive scalar fields in a pure $(n+1)$-dimensional dS background, while appendix B considers the stability of dS space with respect to perturbations by the scalar field modes.

2. Generalities of the dS/CFT correspondence

We wish to discuss the interpretation of scalar field theory in a de Sitter background in the context of the dS/CFT correspondence. However, much of the motivation for the dS/CFT duality, as well as the interpretation of the dS space calculations, comes from our understanding of the AdS/CFT duality [3]. Hence we begin with a brief review of the latter correspondence (section 2.1) and then present the most relevant aspects of dS/CFT (section 2.2). Much of this discussion is review, although our emphasis is somewhat different than in previous treatments. The reader interested in the mathematical details can consult the appendices to which we will refer in the following.

2.1 Brief AdS/CFT review

Consider probing anti-de Sitter space with a massive scalar field. We consider the following

[2]For simplicity, our description is restricted to spherically symmetric foliations [14]. Generically the converging light rays would not be focussed to a single point.

metric on $(n+1)$-dimensional AdS space,[3]

$$ds^2 = dr^2 + e^{2r/\tilde{\ell}}\eta_{\mu\nu}dx^\mu dx^\nu \ , \tag{2.1}$$

and the standard equation of motion for the scalar,

$$\left[\Box - M^2\right]\phi = 0\,. \tag{2.2}$$

Then to leading order in the asymptotic region $r \to \infty$, the two independent solutions take the form [18]

$$\phi_\pm \simeq e^{-\Delta_\pm r/\tilde{\ell}}\phi_{0\pm}(x^\mu) \qquad \text{where} \qquad \Delta_\pm = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + M^2\tilde{\ell}^2}\,. \tag{2.3}$$

Now the interpretation of these results depends on the value of the mass, and there are three regimes of interest:

$$\text{(i)}\ \ M^2 > 0\,, \qquad \text{(ii)}\ \ 0 > M^2 > -\frac{n^2}{4\tilde{\ell}^2} \qquad \text{and} \qquad \text{(iii)}\ \ M^2 < -\frac{n^2}{4\tilde{\ell}^2}\,. \tag{2.4}$$

In case (i), Δ_- is negative and so the corresponding "perturbation" is actually divergent in the asymptotic regime. Hence in constructing a quantum field theory on AdS space, only the ϕ_+ modes would be useful for the construction of an orthogonal basis of normalizable mode functions [19]. In particular, the bulk scalar wave operator is essentially self-adjoint and picks out the boundary condition that the ϕ_- modes are not excited dynamically. In the context of the AdS/CFT then, the ϕ_{0-} functions are associated with source currents (of dimension Δ_-). These may then be used to generate correlation functions of the dual CFT operator of dimension Δ_+ through the equivalence [18, 20]

$$Z_{\rm AdS}(\phi) = \int D\phi\, e^{iI_{\rm AdS}(\phi_-,\phi_+)} = \left\langle e^{i\int \phi_{0-}\mathcal{O}_-}\right\rangle_{\rm CFT}\,. \tag{2.5}$$

On the other hand, the boundary functions ϕ_{0+} are associated with the expectation value for states where the dual operator has been excited [19].

In case (ii), the lower limit corresponds precisely to the Breitenlohner-Freedman bound [21, 22]. While the scalar appears tachyonic, it is not truly unstable and it is still possible to construct a unitary quantum field theory on AdS space. Further, in this regime, both sets of solutions (2.3) are well-behaved in the asymptotic region. However, together they would form an over-complete set of modes. The theory must therefore be supplemented with a boundary condition at AdS infinity which selects out one set of modes to define a self-adjoint extension of the scalar wave operator (and thus the time evolution operator). For $0 > M^2\tilde{\ell}^2 > 1 - n^2/4$, there is a unique boundary condition which produces an AdS invariant quantization [21]. However, for

$$1 - \frac{n^2}{4} > M^2\tilde{\ell}^2 > -\frac{n^2}{4}\,, \tag{2.6}$$

[3]The essential feature for the following analysis is the exponential expansion of the radial slices with proper distance r. While we have chosen to consider pure AdS space in Poincaré coordinates for specificity, this expansion, of course, arises quite generally in the asymptotic large-radius region for any choice of boundary metric and for any asymptotically AdS spacetime.

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the boundary condition is ambiguous. The AdS/CFT interpretation is essentially the same as above. That is, the ϕ_{0+} and ϕ_{0-} functions may be associated with expectation values and source currents of the dual CFT operator, respectively. For the ambiguous regime (2.6), there is a freedom in this equivalence associated with a Legendre transformation of the generating functional [23].

Finally in case (iii), the mass exceeds the Breitenlohner-Freedman bound [21, 22] and the scalar field is actually unstable; no sensible quantization is possible. However, if one were to attempt an AdS/CFT interpretation analogous to those above, the dimension Δ_+ of the dual CFT operator would be complex, which might be interpreted as indicating that the corresponding theory is not unitary. Hence one still seems to have agreement on both sides of the correspondence as to the unsuitability of the regime $M^2\tilde{\ell}^2 < -n^2/4$.

2.2 Some dS/CFT basics

Given the brief overview of the AdS/CFT correspondence, we now turn to asymptotically de Sitter spaces, where one would like to study the possibility of a similar duality between quantum gravity in the bulk and a euclidean CFT [4]. As in the previous review, we focus the present discussion on the case of a pure de Sitter space background:

$$ds^2 = -dt^2 + \cosh^2\frac{t}{\ell}\, d\Omega_n^2\,, \tag{2.7}$$

where $d\Omega_n^2$ is the standard round metric on an n-sphere. This metric solves Einstein's equations, $R_{ij} = 2\Lambda/(n-1)\, g_{ij}$, in $n{+}1$ dimensions. The curvature scale ℓ is related to the cosmological constant by $\ell^2 = n(n-1)/(2\Lambda)$. Again the important feature of this geometry is the exponential expansion in the spatial metric in the asymptotic regions, i.e., $t \to \pm\infty$. Much of the following discussion carries over to spacetimes that only resemble dS asymptotically[4] and indeed, if the proposed dS/CFT duality is to be useful, it must extend to such spacetimes. We will explore certain aspects of the dS/CFT for such backgrounds in section 3.

Consider a free scalar field propagating on the above background (2.7), which we wish to treat in a perturbative regime where the self-gravity is small. Hence, the equation of motion is

$$\left[\Box - M^2\right]\phi = 0\,. \tag{2.8}$$

In general, the effective mass may receive a contribution from a nonminimal coupling to the gravitational field [24]. Therefore we write

$$M^2 = m^2 + \xi R\,, \tag{2.9}$$

where m^2 is the mass squared of the field in the flat space limit and ξ is the dimensionless constant determining the scalar field's coupling to the Ricci scalar, R. In the dS background (2.7), we have $R = n(n+1)/\ell^2$. A case of particular interest in the following section will be that of the conformally coupled massless scalar field, for which $m^2 = 0$, $\xi = (n-1)/4n$ and hence $M^2 = (n^2-1)/4\ell^2$. With these parameters, the solutions

[4]Many explicit examples of such backgrounds may be found in ref. [14].

of eq. (2.8) transform in a simple way under local conformal scalings of the background metric [24].

In parallel with the AdS case, scalar fields propagating in de Sitter space can have two possible behaviors near the boundaries. Let us for the moment think of defining these boundary conditions at past infinity (I^-). Equation (2.8) above is readily solved [4] near I^- to yield two independent solutions with the asymptotic form $\phi \sim e^{h_\pm t/\ell}$ where

$$h_\pm = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - M^2\ell^2}\,. \tag{2.10}$$

Note that this asymptotic time dependence is independent of the details of the spatial mode. In the pure dS background (2.7), the same exponents also govern the behavior of the fields at future infinity — see appendix A for details.

The fact that the boundaries are spacelike in de Sitter space means that the 'boundary conditions' have a different conceptual status than in the AdS setting. In particular, requiring that the bulk evolution is well-defined in dS space will not impose any restrictions on past or future boundary conditions. So in contrast to the AdS/CFT correspondence, in the dS/CFT correspondence, both the ϕ_+ and ϕ_- modes appear on an equal footing.[5] Certainly, a complete description of physics in the bulk must include both sets of modes as dynamical quantum fields. Following the analogy with the AdS/CFT correspondence and in accord with the preceding discussion, it is natural then to associate both modes $\phi_\pm$ with source currents for dual field theory operators $\mathcal{O}_\pm$, with conformal dimensions $h_\mp$ [4]. As we will discuss shortly, this matching of modes with dual operators is further supported by a bulk construction of a generating functional for correlation functions in the CFT.

As in the AdS case, one can classify the scalars displaying distinct types of boundary behavior in three different regimes:

$$\text{(i)}\ \ M^2 > \frac{n^2}{4\ell^2}\,, \qquad \text{(ii)}\ \ \frac{n^2}{4\ell^2} > M^2 > 0 \qquad \text{and} \qquad \text{(iii)}\ \ M^2 < 0\,. \tag{2.11}$$

These three regimes also appear in discussions in the mathematics literature — see, e.g., [25]–[28]. There the scalar field is classified according to M^2 regarded as its $\mathrm{SO}(n+1,1)$ Casimir. A common nomenclature for the three possibilities delineated above is the (i) principal, (ii) complementary (or supplementary) and (iii) discrete series of representations of $\mathrm{SO}(n+1,1)$. As is evident from eq. (2.10), the distinguishing feature of scalar fields in the principal series is that they are oscillatory near past (or future) infinity. In contrast, the exponents for fields in the complementary series are real and positive, and so their asymptotic behavior is a purely exponential damping near both boundaries.

Let us consider case (iii) $M^2 < 0$ in detail. While $h_\pm$ are both real, the modes $\phi_- \sim e^{h_- t}$ diverge as one approaches I^- since $h_- < 0$. One finds similar divergent behavior for one of the modes at the future boundary I^+. The discrete series then corresponds to special values of the mass in this range where a subset of the modes display the convergent h_+

[5] Attempts have been made to distinguish these modes through energy considerations [16] but we disagree with their discussion, as described in appendix B.

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behavior at both $I^{\pm}$ — see appendix A and refs. [25]–[28]. However, we emphasize that even in these special cases, the full space of solutions still includes modes diverging at both asymptotic boundaries. In a physical situation then, the uncertainty principle would not allow us to simply set the amplitude of the divergent modes to zero. Hence the formal mathematical analysis of these fields is only of limited physical interest and we will not consider them further in the following. Of course, the divergence of the generic field configuration is simply an indication that treating the tachyonic fields as linearized perturbations is inappropriate. Nonlinear field theories with potentials including unstable critical points may play an important role in the dS/CFT correspondence, e.g., in constructing models of inflationary cosmology. The essential point though is that one must study the full nonlinear evolution of such fields including their backreaction on the spacetime geometry.

Considering the principal series, (i) $M^2 > n^2/4\ell^2$, in more detail, one expects to find a pair of dual operators $\mathcal{O}_{\pm}$ with *complex* conformal dimensions $h_{\mp}$. Having operators with a complex conformal weight suggests that the dual CFT is nonunitary [4]. We add the brief observation that, since in the quantum field theory, the two sets of independent modes $\phi_{\pm}$ correspond roughly to creation and annihilation operators (positive and negative frequency modes) in the bulk (see, e.g., [29]–[32]), the corresponding operators in the dual theory should have nontrivial commutation relations.

Finally we consider the complementary series, (ii) $n^2/4\ell^2 > M^2 \geq 0$. In this case, the time dependence for both sets of modes is purely a real exponential decay near both $I^{\pm}$. In the bulk, the two linearly independent solutions may be chosen to be real, as is readily verified by explicit computations — see appendix A. Because the $\phi_{\pm}$ solutions are real, they each have zero norm in the usual Klein-Gordon inner product, while a nonvanishing inner product arises from (ϕ_+, ϕ_-). It follows that upon quantization the corresponding operator coefficients are analogous to position and momentum operators, rather than creation and annihilation operators. That is, these degrees of freedom are canonically conjugate. In any event, both types of modes are again required to describe standard quantum field theory in the bulk.

As before, the dual CFT should contain a pair of operators $\mathcal{O}_{\pm}$ dual to the $h_{\mp}$ modes. In this case, the operators have real conformal weights and must be distinct as their weights are different. One can readily see that both $\mathcal{O}_{\pm}$ will have local correlation functions. One simply notes that the corresponding source currents are obtained from the bulk scalar field through

$$\begin{aligned} J_-(\Omega) &\equiv \lim_{t\to-\infty} e^{-h_- t/\ell} \phi(\Omega, t)\,, \\ J_+(\Omega) &\equiv \lim_{t\to-\infty} e^{-h_+ t/\ell} \left[\phi(\Omega, t) - e^{h_- t/\ell} J_-(\Omega)\right], \end{aligned} \tag{2.12}$$

where Ω denotes a point on the n-sphere. As these constructions are local in position, their two-point functions will also be local. Note that the above discussion of inner products indicates that the operators $\mathcal{O}_{\pm}$ should have nontrivial commutation relations with each other but vanishing commutators amongst themselves.

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Next we consider the generator of correlation functions in the dual field theory. A natural construction proposed in [4] for a free bulk field theory is

$$\mathcal{F} = \lim_{t,t'\to-\infty} \int d\Sigma^\mu d\Sigma'^\nu \phi(x) \overleftrightarrow{\partial}_\mu G(x,x') \overleftrightarrow{\partial}_\nu \phi(x') \,. \tag{2.13}$$

In the original proposal of [4], $G(x,x')$ was chosen as the Hadamard two-point function

$$G(x,x') = \langle 0|\{\phi(x),\phi(x')\}|0\rangle \tag{2.14}$$

in the euclidean vacuum, which is symmetric in its arguments. Generalizing this construction to other two-point functions was considered in [29, 33]. These alternatives all provide essentially the same short distance singularities discussed below.

One proceeds by evaluating the generating functional $\mathcal{F}$. First the boundary conditions (2.10) at I^- yield

$$\lim_{t\to-\infty} \phi(\Omega,t) \simeq \phi_{0+}(\Omega)\, e^{h_+ t} + \phi_{0-}(\Omega)\, e^{h_- t} \,, \tag{2.15}$$

where we imagine that $M^2 > 0$ so that the above shows no divergent behavior. Now the dS-invariant two-point function may also be expanded in the limit that $t,t' \to -\infty$ with the result being

$$G(x,x') \simeq c_+ \frac{e^{-h_+(t+t')}}{(w^i w'^i - 1)^{h_+}} + c_- \frac{e^{-h_-(t+t')}}{(w^i w'^i - 1)^{h_-}} \,, \tag{2.16}$$

where c_+ and c_- are constants and w^i denote direction cosines on S^n. Using the notation of [33], one has $w^1 = \cos\theta_1$, $w^2 = \sin\theta_1 \cos\theta_2, \ldots$, $w^d = \sin\theta_1 \cdots \sin\theta_{n-1} \sin\theta_n$. Note in particular that with this choice of coordinates when the points on sphere coincide, one has $w^i w'^i = 1$, while for antipodal points, one has $w^i w'^i = -1$. Taking into account the measure factors, the final result for the generating functional reduces to

$$\mathcal{F} = -\frac{(h_+ - h_-)^2}{2^{2n}} \int d\Omega d\Omega' \left[c_+ \frac{\phi_{0-}(\Omega)\, \phi_{0-}(\Omega')}{(w^i w'^i - 1)^{h_+}} + c_- \frac{\phi_{0+}(\Omega)\, \phi_{0+}(\Omega')}{(w^i w'^i - 1)^{h_-}} \right] . \tag{2.17}$$

Note that the Klein-Gordon inner product has eliminated the cross-terms (which were potentially divergent). Further the coincidence singularities in eq. (2.17) are proportional to the euclidean two-point function on a n-sphere, i.e.,

$$\Delta_{h_\pm} \simeq \frac{1}{(w^i w'^i - 1)^{h_\pm}} \,, \tag{2.18}$$

for operators with conformal weight $h_\pm$. Hence $\mathcal{F}$ appears to be a generating functional for CFT correlation functions with $\phi_{0\pm}$ acting as source currents for operators with conformal dimensions $h_\mp$. The above relies on having a free field theory in the bulk dS space, but extending the construction to an interacting field theory was considered in [33].

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3. CFT on two boundaries

As remarked in the introduction, de Sitter space has two conformal boundaries and so one may ask the question as to whether the dS/CFT correspondence involves a single dual field theory or two. One simple argument in favor of one CFT is as follows [34]: The isometry group of $(n+1)$-dimensional dS space is $\mathrm{SO}(n+1,1)$, which agrees with the symmetries of a single euclidean CFT in n dimensions. Further note that the global Killing vector fields corresponding to these isometries in dS space act nontrivially on both $I^{\pm}$. Hence there is a simple correlated action on source currents or dual operators identified with each of the boundaries. Hence given the single symmetry group, it is natural to think that the dual description involves a single CFT.

Further we would recall our experience from the AdS/CFT correspondence. A central point in this context is that the CFT does not 'live' on the boundary of the AdS space. Usually one has chosen a particular foliation of AdS [35], and the bulk space calculations are naturally compared to those for the field theory living on the geometry of the surfaces comprising this foliation. Via the UV/IR correspondence, each surface in the bulk foliation is naturally associated with degrees of freedom in the CFT at a particular energy scale [36]. The boundary of AdS space plays a special role in calculations as this is a region of the geometry where the separation between operator insertions and expectation values is particularly simple. One notable exception where two CFT's seem to play a role is the eternal black hole [37, 38, 39]. In this case, however, the bulk geometry has two causally disconnected boundaries. In fact, one can show that for any solution of Einstein's equations with more than one asymptotically AdS boundary, the boundaries are all causally disconnected from each other [40]. In the case of dS space, the past and future boundaries are certainly causally connected and so it seems $I^{\pm}$ can be considered as two (special) slices in a certain foliation (2.7) of the spacetime. Hence this reasoning suggests that one should only consider a single CFT in the dual description.

3.1 Nonlocality in the boundary map

Next we turn to Strominger's observation [4] that the generating functional (2.13) can in certain circumstances be extended to incorporate sources on I^{+}. Certainly the construction of the generating functional in the previous section produces essentially the same result if we replace both of the limits in eq. (2.13) with $t, t' \to +\infty$. This would produce an analogous generating functional with source currents defined by the asymptotic behavior of the scalar near I^{+}, i.e.,

$$\lim_{t\to+\infty} \phi(\Omega,t) \simeq \tilde{\phi}_{0+}(\Omega)\, e^{-h_+ t} + \tilde{\phi}_{0-}(\Omega)\, e^{-h_- t} \,. \tag{3.1}$$

However, it is also interesting to consider the case where only one of the limits in eq. (2.13) is replaced with one approaching I^{+},

$$\widetilde{\mathcal{F}} = \lim_{t\to+\infty, t'\to-\infty} \int d\Sigma^{\mu} d\Sigma'^{\nu} \phi(x) \overleftrightarrow{\partial}_{\mu} G(x,x') \overleftrightarrow{\partial}_{\nu} \phi(x') \,. \tag{3.2}$$

Now an essential observation [4, 5] is the causal connection between points on the two boundaries $I^{\pm}$. In particular, a null geodesics emerging from a point on I^{-} expands out into the dS spacetime and refocuses precisely at the antipodal point on the n-sphere at I^+. Hence the two-point function in eq. (3.2) (or any dS-invariant Green's function) will introduce singularities when the point on I^+ approaches the antipode to the point on I^-, as the proper separation of these points vanishes. In fact, in certain circumstances (see the details below), evaluating the above expression yields the simple result:

$$\widetilde{\mathcal{F}} = -\frac{(h_+ - h_-)^2}{2^{2n}} \int d\Omega d\Omega' \left[\tilde{c}_+ \frac{\tilde{\phi}_{0-}(\Omega)\, \phi_{0-}(\Omega')}{(w^i w'^i + 1)^{h_+}} + \tilde{c}_- \frac{\tilde{\phi}_{0+}(\Omega)\, \phi_{0+}(\Omega')}{(w^i w'^i + 1)^{h_-}} \right]. \tag{3.3}$$

This expression incorporates the same euclidean two-point function except that the singularities now arise as the sources $\tilde{\phi}_{0\pm}(\Omega)$ approach antipodes on the n-sphere.

These results suggest that one need only consider a single copy of the CFT and that an operator on I^+ is identified with the same operator on I^- after an antipodal mapping. One finds further support for this interpretation by considering the isometries of dS space. For example, the isometry[6] which produces a dilatation around a point on I^-. On I^+, the same symmetry corresponds to a dilatation around the antipodal point on the n-sphere.

However, this suggestion for identifying operators at I^+ and I^- is easily seen to require some revision as follows. As discussed in the introduction, bulk correlators are naturally related by time evolution. The key ingredient is simply the free field evolution of the scalar, which given some configuration specified on a n-dimensional hypersurface is characterized by the formula

$$\phi(x') = \int d\Sigma^\mu \; \phi(x) \overset{\leftrightarrow}{\partial}_\mu G_R(x, x'), \tag{3.4}$$

where $G_R(x,x')$ is a retarded Green's function, i.e., it vanishes for $t > t'$. Now as an example, the integral appearing in the generating functional (2.13) is covariant and so should be invariant when evaluated on any time slices t and t'. The advantage of pushing these slices to I^- (or I^+) lies in the fact that one can easily separate the source currents according to their conformal weights.

We can explicitly consider the relation between currents on the past and future boundaries by simply following the classical evolution (3.4) of the fields from I^- to I^+. Unfortunately, it is clear that generically there is no simple local relation between the currents on I^- and those on I^+. This remark comes from the observation that in general the retarded Green's function will have support throughout the interior of the light cone. This intuition is readily confirmed by explicit calculations. Ref. [25] presents explicit Green's functions for generic masses in four-dimensional de Sitter space. So, for example, for scalar fields in the principal series, the retarded Green's function becomes for large timelike proper separation

$$G_R(t, \Omega; t', \Omega') \propto \frac{\sin^{n/2}\tau \, \sin^{n/2}\tau'}{(w^i w'^i - \cos\tau \, \cos\tau')^{n/2}} \, \theta(\tau' - \tau), \tag{3.5}$$

where τ is the conformal time coordinate, $\sin\tau = 1/\cosh t/\ell$ — see eq. (3.10), below. Here the θ-function ensures the proper time-ordering of the points. In any event, eq. (3.5) illustrates how the field 'leaks' into the interior of the lightcone with the classical evolution.

[6]This isometry corresponds to the action of a time translation ∂_t in the static patch coordinates [41].

Generically this leads to a nonlocal mapping between the currents on I^- and I^+. This complication will only be avoided in certain exceptional cases, for example, if the retarded Green's function has only support precisely on the light cone — a point to which we return below.

The nonlocal relation between the currents on I^- and those on I^+ can be made more explicit through the mode expansion of the fields on dS space — see appendix A. A well-documented feature of cosmological spacetimes is mode-mixing or particle creation [24]. For the present case of dS space, this corresponds to the fact that a mode of the scalar field with a given boundary behavior on I^-, e.g., having h_- scaling, will usually have a mixture of $h_\pm$ scaling components at I^+. Appendix A provides a detailed discussion of the mode expansions on dS space as well as the Bogolubov transformation relating the modes with a simple time dependence (scaling behavior) near I^- to those near I^+. Using these results, we may discuss the mapping between the currents on the conformal boundaries. Following the notation of appendix A, we decompose the asymptotic fields in terms of spherical harmonics on the n-sphere

$$\phi_{0\pm}(\Omega) = \sum_{L,j} a_{\pm Lj} Y_{Lj}\,, \qquad \tilde{\phi}_{0\pm}(\Omega) = \sum_{L,j} \tilde{a}_{\pm Lj} Y_{Lj}\,. \tag{3.6}$$

Denoting the antipodal map on the n-sphere as $\Omega \to J\Omega$, one has[7] $Y_{Lj}(\Omega) = (-)^L\, Y_{Lj}(J\Omega)$. Now let us imagine that $\phi_{0\pm}$ and $\tilde{\phi}_{0\pm}$ are related by the antipodal map, i.e., $\phi_{0\pm}(\Omega) = z\,\tilde{\phi}_{0\pm}(J\Omega)$ with some constant phase z. Then one must have

$$a_{\pm Lj} = z\,(-)^L\,\tilde{a}_{\pm Lj}\,, \tag{3.7}$$

where in particular the constant z is independent of L.

However, in general, the Bogolubov transformation given in appendix A gives a more complex mapping. For example, from eq. (A.16) for the principal series, one finds

$$a_{\pm Lj} = C_-^-(\omega) e^{\pm 2i\theta_L} \tilde{a}_{\pm Lj} + C_-^+(\omega) \tilde{a}_{\mp Lj}\,. \tag{3.8}$$

Now given eq. (A.17) for n odd with both $C_-^-(\omega)$ and $C_-^+(\omega)$ nonvanishing, certainly eq. (3.7) is inapplicable. One comes closer to realizing the desired result with even n for which $C_-^-(\omega) = 1$ and $C_-^+(\omega) = 0$. However, for either n odd or even, the phase θ_L always introduces a nontrivial L dependence (beyond the desired $(-)^L$) as shown in eq. (A.18). Thus while the mapping between I^- and I^+ may look relatively simple in this mode expansion, it will clearly be nonlocal when expressed in terms of the boundary data $\phi_{0\pm}(\Omega)$ and $\tilde{\phi}_{0\pm}(\Omega)$.

The complementary series gives some more interesting possibilities with

$$\begin{aligned} a_{-Lj} &= \bar{C}_-^-(\mu)\,\tilde{a}_{-Lj} + \bar{C}_-^+(\mu)\,\tilde{a}_{+Lj}\,, \\ a_{+Lj} &= \bar{C}_+^-(\mu)\,\tilde{a}_{-Lj} + \bar{C}_+^+(\mu)\,\tilde{a}_{+Lj}\,. \end{aligned} \tag{3.9}$$

[7]This result becomes clear when the n-sphere is embedded in R^{n+1} with $(x^1)^2+(x^2)^2+\cdots+(x^{n+1})^2 = 1$. In this case, the spherical harmonics Y_{Lj} may be represented in terms of symmetric traceless tensors, $Z_{i_1 i_2\cdots i_L} x^{i_1} x^{i_2} \cdots x^{i_L}$, and hence it is clear that the antipodal map, which takes the form $J : x^i \to -x^i$, produces an overall factor of $(-)^L$.

In particular for n odd and μ half integer, one finds $\bar{C}^{+}_{-}(\mu) = 0 = \bar{C}^{-}_{+}(\mu)$ and $\bar{C}^{-}_{-}(\mu) = (-)^{n/2+\mu}(-)^{L} = \bar{C}^{+}_{+}(\mu)$. Note that these special cases include $\mu = 1/2$, which corresponds to the conformally coupled massless scalar field to which we will return in the following section. Similarly for n even and μ integer: $\bar{C}^{-}_{-} = \bar{C}^{+}_{+} = (-1)^{n/2+\mu+1}(-1)^{L}$ and $\bar{C}^{+}_{-} = 0 = \bar{C}^{-}_{+}$. Hence the coefficients for these special cases give a precise realization of eq. (3.7). Further for these cases then, the generating functional considered in eq. (3.2) will take the simple form given in eq. (3.3).

Hence when considering the principal series or generic masses in the complementary series, it seems that nonlocality will be an unavoidable aspect of the relation between field theory operators associated with two conformal boundaries. The essential point is that the time evolution of the scalar generically introduces nonlocality in the mapping because the retarded Green's function smears a point-like source on I^- out over a finite region on I^+. However, note that one reproduces precisely the same boundary correlators but after some nonlocal reorganization of the degrees of freedom within the dual field theory. It seems appropriate to refer to such relations as *nonlocal dualities* within the field theory. On the other hand, the complementary series does seem to provide some situations where the mapping of the boundary data between I^- and I^+ is local. In the absence of a working example of the proposed dS/CFT duality, one might interpret these results as a hint towards the specific types of fields that would appear in a successful realization of the dS/CFT. Unfortunately, however, this selection rule based on locality of the mapping between boundaries does not seem to survive in more interesting applications, as we will see in the following.

3.2 Nonlocal dualities in 'tall' spacetimes

It is of interest to extend the application of the dS/CFT correspondence from dS space to more general spacetimes with asymptotically dS regions. As a consequence of a theorem of Gao and Wald [17], such a (nonsingular) background will be 'tall' [14]. That is, the conformal diagram for such spacetimes must be taller in the timelike direction than it is wide in the spacelike direction. Of course, this feature has important implications for the causal connection between the past and future boundaries, and hence for the relation between the dual field theory operators defined at these surfaces. In particular, the latter relation becomes manifestly nonlocal.

We may explicitly illustrate the causal structure of the tall spacetimes by working in conformal coordinates. For asymptotically dS spacetimes which are homogeneous on spherical hypersurfaces, the metric may be written

$$ds^2 = C(\tau)\left[-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega^2_{n-1}\right]. \tag{3.10}$$

For pure dS space, $C(\tau) = \ell^2/\sin^2\tau$. Note that the proper time t in eq. (2.7) is related to the conformal time τ above by the coordinate transformation $\sin\tau = 1/\cosh t/\ell$.

In this case, the conformal time runs from $\tau = 0$ at I^- to $\tau = \pi$ at I^+. The angular coordinate θ on the n-sphere runs over the same range, i.e., from $\theta = 0$ at the north pole to $\theta = \pi$ at the south pole. Hence it is clear that the conformal diagram for dS space is a square (see figure 1).

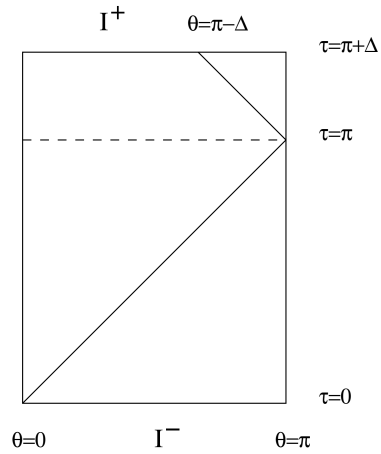

Figure 2: Conformal diagrams of a perturbed de Sitter space. The excess height is represented by Δ.

Now for a tall spacetime, the conformal time above would run over an extended range $0 \leq \tau \leq \pi + \Delta$ where $\Delta > 0$. The assumption that the background is asymptotically dS means that the conformal factor has the following behavior near $I^{\pm}$:

$$\begin{aligned}
\lim_{\tau \to 0} C(t) &= \frac{\ell^2}{\sin^2 \tau}\,, \\
\lim_{\tau \to \pi+\Delta} C(t) &= \frac{\tilde{\ell}^2}{\sin^2(\tau - \Delta)}\,,
\end{aligned} \tag{3.11}$$

where we have allowed for the possibility that the cosmological 'constant' is different at I^+ than at I^-. This possibility may be realized in a model where a scalar field rolls from one critical point of its potential to another [14]. In any event, the corresponding conformal diagram will be a rectangle with height $\delta\tau = \pi + \Delta$ and width $\delta\theta = \pi$ (see figure 2).

This increase in the height of the conformal diagram modifies the causal connection between $I^{\pm}$ in an essential way. Consider the null rays emerging from the north pole ($\theta = 0$) at I^- ($\tau = 0$). This null cone expands out across the n-sphere reaching the equator ($\theta = \pi/2$) at $\tau = \pi/2$, and then begins to reconverge as it passes into the southern hemisphere. The null rays focus at the south pole ($\theta = \pi$) at $\tau = \pi$, however, in this tall spacetime, this event corresponds to a finite proper time for an observer at the south pole. Beyond this point, the null cone expands again and intersects I^+ ($\tau = \pi + \Delta$) on the finite-sized (n–1)-sphere at $\theta = \pi - \Delta$.

The discussion of the previous section made clear that an essential ingredient in finding a simple local mapping of boundary data on I^- to that on I^+ in dS space was the refocusing of the above null cone precisely at the future boundary. Even in that case, we pointed out that the time evolution of the scalar generically introduces nonlocality in the mapping because the retarded Green's function smears a point-like source on I^- out over a finite region on I^+. Here we see that in a tall spacetime, a nonlocal map is inevitable since the causal connection between the past and future boundaries is itself nonlocal. So we should expect that even in the special cases found to have a local map for pure dS space, the mapping should become nonlocal for these same theories in a tall background. That is, for these more general asymptotically dS spacetimes, the relation between the dual field theory operators defined at each of the boundaries becomes nonlocal. Hence we are naturally lead to consider a nonlocal self-duality of the CFT. Further we note that given the results of Gao and Wald [17] this would be the generic situation. For example, injecting a single scalar field quantum into dS space would actually lead to backreaction effects which would produce a tall spacetime.

3.3 The conformally coupled massless scalar

We now turn to consider conformally coupled massless scalar field theory as an example which illustrates several of the points discussed above. In particular, it is an example where the mapping between the past and future boundaries is local in pure dS space, but becomes nonlocal in a tall background. Another useful feature is that one can perform explicit calculations in a tall spacetime without referring to the detailed evolution of the conformal factor $C(\tau)$. Rather a knowledge of the boundary conditions (3.11) is sufficient.

The conformally coupled massless scalar corresponds to the curvature coupling $\xi = \frac{n-1}{4n}$ and $m^2 = 0$ in eq. (2.9). Hence in pure dS space or in an asymptotically dS region, $M^2\ell^2 = (n^2-1)/4$ and the corresponding scaling exponents (2.10) become $h_\pm = (n\pm 1)/2$, independent of the value of the cosmological constant. As one might infer from the real exponents, this field lies in the complementary series for any value of the cosmological constant. The remarkable property of this scalar field theory is that the solutions of the wave equation (2.8) transform in a simple way under local conformal scalings of the background metric [24].

The backgrounds of interest (3.10) are conformally flat[8] and therefore the Green's function describing the evolution in the tall background is simply the flat space Green's function for a *massless* scalar field, up to some overall time dependent factors. In particular then, for d even (n odd), the Green's function will have support precisely on the light cone. For example, in four-dimensional dS space, the retarded Green's function can be written as

$$G_R(\tau,\Omega;\tau',\Omega') = -\frac{\sin\tau\sin\tau'}{4\pi\ell^2}\,\delta(w^i w'^i - \cos(\tau'-\tau))\,\theta(\tau'-\tau)\,. \tag{3.12}$$

Similarly in higher even-dimensional dS spaces, the Green's function will contain δ-functions (and derivatives of δ-functions) with support only on the light cone [42]. Given this form of the retarded Green's functions, the evolution of the scalar field (3.4) from I^- to I^+ will produce precisely the antipodal mapping for all of these cases. Note that this result is confirmed by the mode analysis in the first part of this section. The conformally coupled massless scalar has $\mu = 1/2$ and we are considering even dimensions or n odd. This combination matches one of the special cases in which the modes transformed according to the antipodal mapping.

Using the conformal transformation properties of the field [24], the analogous Green's function for any spacetime of the form (3.10) is easily constructed. For $d = 4$, it may be

[8] Note that the coordinate transformation $T = e^\tau$ puts the metric (3.10) in the form of the flat Milne universe, up to a conformal factor.

written as

$$G_R(\tau,\Omega;\tau',\Omega') = -\frac{1}{4\pi}\frac{1}{\sqrt{C(\tau)}\sqrt{C(\tau')}}\,\delta(w^i w'^i - \cos(\tau'-\tau))\,\theta(\tau'-\tau)\,. \tag{3.13}$$

For other values of even d, the corresponding Green's function has a similar form. For the conformally coupled scalar in such tall spaces, the delocalization of the boundary map does not depend on the detailed evolution, i.e., the details of $C(\tau)$. Rather the nonlocality is completely characterized by Δ, the excess in the range of the conformal time. For example, a source current placed at the north pole ($\theta = 0$) on I^- is smeared over an (n–1)-sphere centered at the south pole ($\theta = \pi$) and of angular radius $\delta\theta = \Delta$ on I^+.

Using eq. (3.13), we can make this discussion completely explicit for four dimensions. Consider an arbitrary tall space (3.10) with $n = 3$ satisfying the boundary conditions given in eq. (3.11). First with the conformal time coordinate, the asymptotic boundary conditions (2.15) for the scalar field at I^- become

$$\lim_{\tau\to 0}\phi(\Omega,\tau) \simeq \phi_{0+}(\Omega)\,\tau^{h_+} + \phi_{0-}(\Omega)\,\tau^{h_-} \tag{3.14}$$

and similarly at I^+, we have

$$\lim_{\tau\to\pi+\Delta}\phi(\Omega,\tau) \simeq \tilde{\phi}_{0+}(\Omega)\,(\pi+\Delta-\tau)^{h_+} + \tilde{\phi}_{0-}(\Omega)\,(\pi+\Delta-\tau)^{h_-}\,. \tag{3.15}$$

These boundary conditions apply for a general scalar field theory. In the present case of a conformally coupled massless scalar with $n = 3$, we have $h_+ = 2$ and $h_- = 1$. Hence inserting (3.13) and (3.14) into (3.4), we may evaluate the result at a point $(\Omega',\tau' = \pi+\Delta-\epsilon)$ near I^+ and compare to eq. (3.15). The final result for the boundary fields on I^+ is

$$\begin{aligned}
\tilde{\phi}_{0+}(\Omega') &= \frac{|\sin\Delta|}{\sin\Delta}\frac{\ell}{\tilde{\ell}}\Big\{\sin\Delta\,\langle\phi_{0-}\rangle_\Delta(J\Omega') - \cos\Delta\,\langle\phi_{0+}\rangle_\Delta(J\Omega')\Big\} \\
\tilde{\phi}_{0-}(\Omega') &= \frac{|\sin\Delta|}{\sin\Delta}\frac{\ell}{\tilde{\ell}}\Big\{\cos\Delta\,\langle\phi_{0-}\rangle_\Delta(J\Omega') + \sin\Delta\,\langle\phi_{0+}\rangle_\Delta(J\Omega') + \sin\Delta\,\partial_\theta\langle\phi_{0-}\rangle_\Delta(J\Omega')\Big\}\,,
\end{aligned} \tag{3.16}$$

where J is the antipodal map on the two-sphere and $\langle\phi_{0\pm}\rangle_\Delta(J\Omega')$ denote the average of $\phi_{0\pm}$ on the two-sphere separated from $J\Omega'$ by an angle Δ. The factors $|\sin\Delta|/\sin\Delta$ are to be understood as being continuous from below; i.e., this factor is -1 at $\Delta = 0$ and $+1$ at $\Delta = \pi$.

This expression simplifies tremendously in the case of dS space with $\Delta = 0$ (as well as $\tilde{\ell} = \ell$) to yield

$$\tilde{\phi}_{0+}(\Omega') = \phi_{0+}(J\Omega')\,, \qquad \tilde{\phi}_{0-}(\Omega') = -\phi_{0-}(J\Omega')\,. \tag{3.17}$$

Thus, in pure four-dimensional dS space the map from I^- to I^+ acts on the conformally coupled massless scalar field as simply the antipodal map on ϕ_{0+} and -1 times the antipodal map on ϕ_{0-}. Note that the time reflection symmetry of de Sitter allows solutions for the mode functions to be decomposed into even and odd parts and, furthermore, both even and odd solutions will exist. Thus, with our conventions and $h_\pm$ real, when evolution from I^- to I^+ leads to the antipodal map it will be associated with a phase $z = +1$ for one set of modes and the opposite phase $z = -1$ for the other.

4. Discussion

The dS/CFT correspondence is a striking proposal which carries the potential for extraordinary new insights into cosmology and the cosmological constant problem. Unfortunately, the outstanding problem remains to find a concrete example where the bulk gravity theory and the dual field theory are understood or at least known. Lacking the guidance that such a working model would provide, one is left to study various aspects of physics in (asymptotically) dS spacetimes from this new point of view and to determine properties which this correspondence implies for the dual euclidean CFT.

Such investigations have yielded a number of unusual properties for the dual field theory. It is likely to be nonunitary, e.g., if the bulk theory involves scalars in the principal series [4]. A nonstandard inner product is required to reproduce ordinary quantum field theory in the bulk [29, 6]. One might also observe that this euclidean field theory should not simply be a standard Wick rotation of a conventional field theory since attempting to 'un-Wick rotate' would produce a bulk theory with two time directions and all of the associated confusions. We may add to this list the observation of section 2.2 that, since bulk correlators are not symmetric in Lorentz signature quantum field theory, a straightforward duality would require non-symmetric correlation functions in the dual euclidean theory. But correlators generated by functional differentiation of a partition function are always symmetric, so the euclidean theory could have no definition through a partition sum. Finally, in the present paper, we have also inferred the existence of unusual nonlocal dualities within the field theory itself.

Our investigation focussed on the mapping of operators between I^+ and I^- provided by time evolution in the bulk spacetime. The essential point is that the time evolution of the scalar generically introduces nonlocality in the mapping because the retarded Green's function smears a point-like source on I^- out over a finite region on I^+. However, despite this nonlocal reorganization of the degrees of freedom within the dual field theory, one reproduces the same boundary correlators. Hence we referred to this relation as a nonlocal duality within the field theory. While this nonlocality already applies for many fields in pure dS space, it seems unavoidable in tall spacetimes because the causal connection between I^+ and I^- is inherently nonlocal. We emphasize that tall spacetimes are quite generic as a result of the theorem in [17]. As soon as one perturbs dS even slightly by, e.g., the introduction of matter fields or gravitational waves, the resulting background solution will have the property that its conformal diagram is taller than it is wide. As the inferred self-duality is nonlocal, i.e., local operators are mapped to nonlocal operators, it seems that the underlying field theory does not have a unique concept of locality. That is, one has a specific dictionary whereby the same short distance singularities can be reproduced by a set of local or nonlocal operators.

Faced with the daunting task of consolidating all of these unusual characteristics in a single euclidean field theory, one is tempted to revise the interpretation of the dS/CFT correspondence. One suggestion [6] is that the duality should involve two CFT's but that dS spacetime is defined as a correlated state in Hilbert space of the two field theories. The correlated state is constructed so as to preserve a single SO$(n+1,1)$ symmetry group, which is then reflected in the isometries of the dS space. As discussed in section 3, we still feel that our experience with the AdS/CFT is highly suggestive that the two boundaries should not be associated with distinct field theories. Further, it is difficult to see how this framework could incorporate big bang or big crunch backgrounds with a single asymptotic dS region. Note that the latter spacetimes will still give rise to horizons, as well as the associated thermal radiation and entropy.

However, this approach with two CFT's remains an intriguing suggestion. Within this context, the mapping of the boundary data between $I^{\pm}$ would provide information about correlations in the field theory state. Hence our calculations would still find application in this context. The nonlocalities discussed here, while not unnatural, give an indication of the complexity of these correlations.

We should also remark that all of our investigations treated only the time evolution of a free scalar field theory. The mapping of boundary operators will become even more complex if one was to consider an interacting field theory. Of course, in accord with the discussion here, we would still expect that time evolution of the fields or operators in an interacting theory would still provide the basis for this mapping.

While it is amusing to speculate on such matters, we note that the central thesis of [43] is that one cannot successfully understand the physics of dS space within the context of quantum field theory in curved spacetime. It is interesting to consider how their comments may relate our discussion. Essentially, they suggest that bulk properties of dS space should be analogous to the physics in a thermal system with a finite number of states and deduce from this that the evolution map of linearized quantum field theory should not be trusted in detail near the past and future boundaries. As a result, they suggest that a dual theory may not be as local as one might expect by studying limits of bulk correlation functions in background quantum field theory. The comments of [44] raise further questions about correlation functions between points with a large separation in time. In particular, the problematic correlators would include precisely those between operators on I^- and I^+. Here the smearing observed in the tall spacetimes is likely to play a role since, if backreaction is properly accounted for, even injecting a single scalar field quantum into dS should deform it to a (slightly) tall spacetime. It may be that the nonlocalities discussed here may be a hint that the 'correct physical observables' are themselves nonlocal[9] so that the boundary map would preserve the form of such operators.

Note that there is a certain tension between our strong reliance on time evolution, through which observables near any two Cauchy surfaces can be related, and the idea that the bulk evolution is related to a renormalization group flow in the dual theory [12, 13]. The point is that time evolution naturally produces a scaling of distances on Cauchy surfaces (at least in simple examples) and so these surfaces are naturally associated with different distance scales in the dual theory. However, the time evolution map relating different surfaces is invertible. In contrast, the usual notion of the renormalization group is actually that of a semi-group, in which different scales are related by integrating out modes, i.e.,

[9] Similar implications can be drawn from the finite time resolution discussed in [45].

by throwing away short distance details so that the descriptions at two different scales are not fully equivalent.

To gain some perspective on this issue, we would like to return briefly to the AdS/CFT case and the interpretation of renormalization group flows. Recall that the primary assumption is that the relevant asymptotically AdS spacetime is in fact dual to the vacuum of some field theory. The important point is that one begins by placing the entire spacetime in correspondence with the vacuum of some single theory. One then uses the IR/UV connection to argue that different regions of the bulk spacetime are naturally related to different energy regimes in the dual theory. The suggestion that this description at differing energy scales is somehow connected to a renormalization group flow seems natural and, in that context, there was no evolution map relating the inner and outer regions to provide such an obvious tension.

In the present dS/CFT context, such a tension does exist. However, the more primitive association of different parts of the spacetime with the behavior of the field theory at differing energy scales still seems plausible. A more concrete version of this idea is suggested by the behavior of the bulk evolution map itself. As we have seen, the evolution map from t to t' 'coarse grains' the observables on t' in the sense that the theory is now presented in terms of variables (those that are local at time t) which are nonlocal averages over the intersection of past light cones from time t with the original hypersurface at t'. However, a sufficient number of overlapping coarse grainings are considered that no information is lost. Such a procedure can also be performed in a euclidean field theory and one might speculate that keeping only the simplest terms in the resulting action might bare some similarity to those obtained from more traditional renormalization group methods. This would be in keeping with the identification of a c-function [12]–[14] in which a spacetime region was associated with a copy of de Sitter space by considering only the metric and extrinsic curvature on a hypersurface.

Note that this interpretation readily allows us to run our flow both 'forward' (toward the IR) and 'backward' (toward the UV). However, it is far from clear that the coarse graining procedure is unique. This fits well with the interpretation suggested in [14] for 'renormalization group flow spacetimes' with spherical homogeneity surfaces. There, one naturally considers two UV regions (one at I^- and one at I^+) which both 'flow' to the same theory at some minimal sphere where the two parts of the spacetime join. One simply reads the flow as starting in the UV, proceeding toward the IR, but then reversing course. Interestingly, it is possible to arrive at a different UV theory from which one began. Such an odd state of affairs is more natural when one recalls that we have already argued that the theory must possess a nonlocal duality, so that it in fact has two distinct local descriptions.

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A. Scalar field modes in dS space

In this appendix, we present a detailed analysis of the bulk physics of massive scalar fields propagating in a dS space of arbitrary dimension, emphasizing characteristics of their evolution which should be relevant to the proposed dS/CFT correspondence. Our aim is to characterize fully the mode mixing phenomenon inherent to physics in dS space. While the details of this analysis are readily available in the literature for the modes of the principal series (see, for example, ref. [29]), we did not find explicit accounts of the complementary and discrete series.

A.1 Field equation and asymptotic behavior

The spherical foliation of $(n{+}1)$-dimensional dS space is given by the metric

$$ds^2 = -dt^2 + \cosh^2 t \; d\Omega_n^2 \,, \tag{A.1}$$

where in this appendix we set the dS radius to unity ($\ell = 1$). We consider a massive scalar field propagating in this background according to

$$\left[\Box - M^2\right] \phi(x) = 0 \,. \tag{A.2}$$

It is convenient to write the solutions to eq. (A.2) in the form

$$\phi(x) = y_L(t) \; Y_{Lj}(\Omega) \,, \tag{A.3}$$

where the Y_{Lj}'s are spherical harmonics on the n-sphere satisfying

$$\nabla^2 Y_{Lj} = -L(L+n-1) Y_{Lj} \,, \tag{A.4}$$

where ∇^2 is the standard laplacian on the n-sphere. The differential equation for $y_L(t)$ is then

$$\ddot{y}_L + n \tanh t \; \dot{y}_L + \left[M^2 + \frac{L(L+n-1)}{\cosh^2 t}\right] y_L = 0 \,. \tag{A.5}$$

As discussed in section 2.2, of particular relevance to the dS/CFT correspondence is the behavior of the scalar field near the boundaries I^+ and I^- as $t \to \pm\infty$. In these limits, eq. (A.5) becomes

$$\ddot{y}_L \pm n \dot{y}_L + M^2 y_L = 0 \,, \tag{A.6}$$

which implies that

$$\lim_{t\to-\infty} y_L \sim e^{h_\pm t} \,, \qquad \lim_{t\to+\infty} y_L \sim e^{-h_\pm t} \,, \tag{A.7}$$

where the weights $h_\pm$ are defined by

$$h_\pm = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - M^2} \equiv \frac{n}{2} \pm \mu \,. \tag{A.8}$$

Formally, one may classify scalar fields according to the irreducible representations of $SO(n+1,1)$, the isometry group of de Sitter space, which are labelled by the eigenvalues associated with the Casimir operator[10] $Q = \Box$, which simply corresponds to the mass parameter M^2. The *principal series* is defined by the inequality $M^2 > n^2/4$. In this case, the weights $h_\pm$ have an imaginary part, and the corresponding modes, while still being damped near the boundaries, have an oscillatory behavior in the bulk. For the *complementary series*, the effective mass falls in the range $0 < M^2 \leq n^2/4$. As will be made more explicit later, the modes are non-oscillatory asymptotically in this case since both $h_\pm$ are real quantities. The remaining *discrete series* corresponds to $M^2 < 0$. This last condition means that $h_- < 0$ (and $h_+ > n$), which implies that the tachyonic fields scaling like $y_L \sim e^{\mp h_- t}$ in approaching on $\mathcal{I}^\pm$ are growing without bound. Still, one is able to find 'normalizable' modes in a certain limited number of cases, as will be discussed below.

The case of $M^2 = 0$, i.e., a massless scalar field, is interesting and deserves further comment. One finds that in this case it is impossible to construct a vacuum state which is invariant under the full de Sitter group $SO(n+1,1)$. A great deal of discussion about the peculiar nature of this quantum field theory can be found in the literature [48, 49, 30]. The weights associated with the $M^2 = 0$ field are $h_+ = n$ and $h_- = 0$. Dual to the latter should be a marginal operator in the CFT, i.e., a deformation which does not scale under conformal transformations.

To fully solve eq. (A.5), we make the change of variables

$$y_L(t) = \cosh^L t \, e^{(L+n/2+\mu)t} g_L(t) \,. \tag{A.9}$$

Setting $\sigma = -e^{2t}$, this equation for the time dependant profile takes the form of the hypergeometric equation:

$$\sigma(1-\sigma)g'' + [c - (1+a+b)\sigma]\, g' - abg = 0 \,, \tag{A.10}$$

where a 'prime' denotes a derivative with respect to σ and the coefficients are

$$a = L + \frac{n}{2} \,, \qquad b = L + \frac{n}{2} + \mu \,, \qquad c = 1 + \mu \,. \tag{A.11}$$

The two independent solutions can then be expressed in terms of hypergeometric functions,

$$y_{L+}(t) = N_+ \cosh^L t \, e^{(L+h_+)t} F\left(L + \frac{n}{2}, L + h_+; 1+\mu; -e^{2t}\right) , \tag{A.12}$$

$$y_{L-}(t) = N_- \cosh^L t \, e^{(L+h_-)t} F\left(L + \frac{n}{2}, L + h_-; 1-\mu; -e^{2t}\right) , \tag{A.13}$$

[10]In fact, there are two coordinate invariant Casimir operators associated with the de Sitter isometry group but only one is relevant in characterizing massive scalar fields. The other Casimir operator automatically vanishes for all spin-zero fields but may play a role in the classification of higher spin representations [46]. Another interesting formal question is the behavior of these representations in the limit where the cosmological constant is taken to zero. A complete treatment of representation contraction in de Sitter space can be found in ref. [47].

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where $N_{\pm}$ are normalization constants, which will be fixed below. More specifically, we have chosen here the two linearly independent solutions of eq. (A.10) in the neighborhood of the singular point $-e^{2t} = 0$ [50], which corresponds to one of the two limits of interest, i.e., $t \to -\infty$. Following eq. (A.3), we denote the complete mode functions as $\phi_{L\pm} = y_{L\pm}(t) Y_{Lj}(\Omega)$.

One important aspect of time evolution of the scalar field in the bulk is the mode mixing that occurs between the two boundaries, $I^{\pm}$. For example, this would be related to the particle production in the dS space [29]–[32]. In the following, we emphasize the differences between the principal, complementary and discrete series.

A.2 Principal series

The principal series is frequently discussed in the physics literature, e.g., [29]–[32], and would seem to be the most relevant case for the particle spectrum observed in nature. We review some of the salient points here for comparison with the other representations in the following subsection. For the principal series, it is useful to introduce $\omega \equiv -i\mu$. Then the above modes become

$$\begin{aligned} y_{L-}(t) &= \frac{2^{L+(n-1)/2}}{\sqrt{\omega}} \cosh^L t \, e^{(L+n/2-i\omega)t} F\left(L+\frac{n}{2}, L+\frac{n}{2}-i\omega; 1-i\omega; -e^{2t}\right), \\ y_{L+}(t) &= \frac{2^{L+(n-1)/2}}{\sqrt{\omega}} \cosh^L t \, e^{(L+n/2+i\omega)t} F\left(L+\frac{n}{2}, L+\frac{n}{2}+i\omega; 1+i\omega; -e^{2t}\right), \end{aligned} \tag{A.14}$$

where $y^*_{L-}(t) = y_{L+}(t)$. Here the normalization constants have been fixed by imposing $(\phi_{L+}, \phi_{L+}) = 1 = (\phi_{L-}, \phi_{L-})$ as usual with the standard Klein-Gordon inner product [24]. As emphasized above, these solutions have the simple time dependence of eq. (A.7) in the asymptotic region $t \to -\infty$ near I^-. Because the differential equation (A.5) is invariant under $t \to -t$, one can easily define another pair of linearly independent solutions by applying this transformation to the above modes. We label the resulting modes: $y_L{}^-(t)$ and $y_L{}^+(t) = y_L{}^{-*}(t)$, where

$$y^-(t) = y^*_+(-t) \,. \tag{A.15}$$

It readily follows that $y_L{}^- \sim e^{-h_- t}$ and $y_L{}^+ \sim e^{-h_+ t}$ near I^+. The two sets of modes $y_{L\pm}(t)$ and $y_L{}^{\pm}(t)$ can respectively be used to construct the 'in' and 'out' vacua with no incoming and outgoing particles. The Bogolubov coefficients relating these two sets of modes are defined through

$$y_{L-}(t) = C^-_-(\omega)\, e^{-2i\theta_L}\, y_L{}^-(t) + C^+_-(\omega)\, y_L{}^+(t)\,, \tag{A.16}$$

with a similar expression for y_{L+} (with $C^+_+(\omega) = C^-_-(\omega)$ and $C^-_+(\omega) = C^+_-(\omega)$). When n is even [29], one finds that $C^-_-(\omega) = 1$ and $C^+_-(\omega) = 0$. This corresponds to the physical statement that there is no particle creation (no mode mixing) in dS space for an odd number of spacetime dimensions. For n odd, there is nontrivial mode mixing with,

$$C^-_-(\omega) = \coth \pi\omega \qquad C^+_-(\omega) = (-1)^{(n+1)/2} \frac{1}{\sinh \pi\omega}\,, \tag{A.17}$$

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where $|C_-^-(\omega)|^2 - |C_-^+(\omega)|^2 = 1$ holds since the modes are properly normalized throughout their evolution. The expression for the phase in eq. (A.16) is

$$e^{-2i\theta_L} = (-1)^{L-n/2}\frac{\Gamma(-i\omega)\Gamma(L+n/2+i\omega)}{\Gamma(i\omega)\Gamma(L+n/2-i\omega)}\,. \tag{A.18}$$

It is clear that for large enough ω, the mixing coefficient $C_-^+(\omega)$ becomes negligible which is in accord with the intuition that there will be limited particle production in high energy modes. We will find that in the other two series there is no phase comparable to eq. (A.18). This complicates the expressions for mode mixing between the boundaries and will lead to interesting features.

A.3 Complementary series and tachyonic fields

For the modes of both the complementary and the tachyonic series, the weights h_+ and h_- are real and so the above mode functions are entirely real,

$$y_{L+}(t) = \bar{N}_+ \cosh^L t\, e^{(L+n/2+\mu)t} F\left(L+\frac{n}{2}, L+\frac{n}{2}+\mu; 1+\mu; -e^{2t}\right), \tag{A.19}$$
$$y_{L-}(t) = \bar{N}_- \cosh^L t\, e^{(L+n/2-\mu)t} F\left(L+\frac{n}{2}, L+\frac{n}{2}-\mu; 1-\mu; -e^{2t}\right). \tag{A.20}$$

Hence with respect to the usual Klein-Gordon product these two solutions have zero norm, i.e., $(\phi_{L+}, \phi_{L+}) = 0 = (\phi_{L-}, \phi_{L-})$.

Of course, this is not unnatural. One gains intuition by considering the usual plane wave decomposition in flat spacetime. There, one may choose between two bases, the one involving complex exponentials and the one involving cosines and sines. The latter basis in fact has the same characteristics as the present modes in the complementary series in terms of normalization with respect to the Klein-Gordon inner product. Consequently, to define a reasonable normalization for the mode functions (A.19) and (A.20), we require $(\phi_{L-}, \phi_{L+}) = i$

$$\bar{N}_+ = \frac{2^{L+(n-1)/2}}{\sqrt{\mu}} = \bar{N}_-\,, \tag{A.21}$$

where we have resolved the remaining ambiguity by simply demanding that $\bar{N}_+ = \bar{N}_-$. With this choice of normalization, it is clear that upon quantizing the scalar field in the dS background the corresponding mode coefficients will have commutation relations analogous to those of coordinate and momentum operators, rather than raising and lowering operators.

As in the previous subsection, by substituting $t \to -t$, we define modes $y_L{}^\pm(t) \equiv y_{L\pm}(-t)$ which have the simple time dependence of eq. (A.7) in the asymptotic region approaching I^+. Using a simple identity of hypergeometric functions [50], one can relate the two sets of modes as

$$\begin{aligned} y_{L-}(t) &= \bar{C}_-^-(\mu)\, y_L{}^-(t) + \bar{C}_-^+(\mu)\, y_L{}^+(t)\,, \\ y_{L+}(t) &= \bar{C}_+^-(\mu)\, y_L{}^-(t) + \bar{C}_+^+(\mu)\, y_L{}^+(t)\,, \end{aligned} \tag{A.22}$$

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where the elements of the mixing matrix C (the Bogolubov coefficients) are given by

$$\bar{C}_{-}^{+}(\mu) = \frac{\Gamma(1-\mu)\Gamma(-\mu)}{\Gamma((2-n)/2-\mu-L)\Gamma((n/2)-\mu+L)}, \qquad \bar{C}_{-}^{-}(\mu) = -(-1)^L \frac{\sin((\pi\, n)/2)}{\sin\pi\mu}, \tag{A.23}$$

$$\bar{C}_{+}^{-}(\mu) = -\frac{\Gamma(1+\mu)\Gamma(\mu)}{\Gamma((2-n)/2+\mu-L)\Gamma(n/2+\mu+L)}, \qquad \bar{C}_{+}^{+}(\mu) = -(-1)^L \frac{\sin((\pi\, n)/2)}{\sin\pi\mu}. \tag{A.24}$$

We now describe some features of the resulting mode mixing for the complementary series. In this case, recall that $0 < M^2 \leq n^2/4$ which implies that $0 \leq \mu < n/2$. Of course, certain features depend on the spacetime dimension $n+1$ as before:

a) *n odd:* generically for the case of an even spacetime dimension, there is nontrivial mode mixing. An exception occurs for $\mu = (2m+1)/2$ with m, a positive integer. For these special cases, there is no mixing since $\bar{C}_{-}^{+} = 0 = \bar{C}_{+}^{-}$ and one finds that $\bar{C}_{-}^{-} = \bar{C}_{+}^{+} = (-1)^{n/2+\mu}(-1)^L$.

b) *n even:* generically for this case of an odd number of spacetime dimensions, one finds $\bar{C}_{+}^{+} = 0 = \bar{C}_{-}^{-}$ and $\bar{C}_{-}^{+}\bar{C}_{+}^{-} = -1$ (where $\bar{C}_{-}^{+}$ and $\bar{C}_{+}^{-}$ both have a nontrivial dependence on L). This means that a mode that is scaling like $e^{h_{\pm}t}$ on I^- will have the 'opposite' scaling $e^{-h_{\mp}t}$ on I^+. We refer to this phenomenon as 'maximal mixing'. This phenomenon is absent when μ is an integer. This case must be treated with some care as the solution for y_{L-} appearing in eq. (A.20) breaks down.[11] The correct solution [50] has an additional logarithmic singularity near I^-, i.e., , subdominant power law behavior in t. In any event, the final result for n even and μ integer is: $\bar{C}_{-}^{-} = \bar{C}_{+}^{+} = (-1)^{n/2+\mu+1}(-1)^L$ and $\bar{C}_{-}^{+} = 0 = \bar{C}_{+}^{-}$.

Finally we briefly consider the tachyonic or discrete series [26]–[28]. Recall that in this case with $M^2 < 0$, $h_- < 0$ so that the modes scaling as $e^{\pm h_- t}$ diverge as one approaches either I^- or I^+, respectively. Generically there is nontrivial mode mixing and so even if a mode is convergent at one asymptotic boundary it will be divergent at the opposite boundary. However, an interesting exceptional case is when a y_{L+} mode (scaling like $e^{h_+ t}$ as $t \to -\infty$) evolves to the corresponding $y_L{}^+$ mode (with $e^{-h_+ t}$ behavior for $t \to \infty$). Such a mode would have convergent behavior both towards the future and past boundaries. This behavior would result when $\bar{C}_{+}^{-}$ vanishes. A brief examination of eq. (A.24) requires that $1 + |h_-| - L$ is zero or a negative integer. As above this constrains μ to be an integer or half-integer depending on the spacetime dimension. We express this constraint in terms of the (tachyonic) mass

$$-M^2 = \begin{cases} \frac{1}{4}((2m+1)^2 - n^2) & \text{for } n \text{ odd} \quad \text{with } m = \frac{(n-1)}{2}, \frac{(n+1)}{2}, \ldots \\ m^2 - \frac{n^2}{4} & \text{for } n \text{ even} \quad \text{with } m = \frac{n}{2}, \frac{n}{2}+1, \ldots \end{cases} \tag{A.25}$$

[11] Similar remarks apply for n odd and μ integer, but in that case one still finds nontrivial mode mixing.

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This constraint gives rise to the nomenclature that these modes are often referred to as the discrete series. However, the above constraint is not sufficient; rather we must also impose a constraint on the 'angular momentum' quantum number L, namely,

$$L \geq 1 + |h_-| \,. \tag{A.26}$$

Hence the completely convergent modes only appear for sufficiently large angular momenta. Note that it is still true that using the usual Klein-Gordon inner product these modes have a vanishing norm $(\phi_{L+}, \phi_{L+}) = 0$. However, in the mathematics literature (e.g., [28]) these modes are singled out by having finite norm in the sense given by the spacetime integral: $\int d^{n+1}x\sqrt{-g}\,|\phi_{L+}|^2 = 1$.

This construction shows that even in the tachyonic mass range, one can find certain normalizable modes (in the above sense) for special choices of parameters. However, we reiterate that while these formal results for the discrete series may be interesting mathematically, they are not useful in understanding the physics of dS space. As emphasized above in the discussion of the dS/CFT correspondence, one must consider the full space of solutions, and presently even in the exceptional cases, the normalizable modes are accompanied by modes diverging at both asymptotic boundaries. Thus, the normalizable modes do not form a complete set of modes on a Cauchy surface. Such divergences, which occur in the generic case as well, are simply an indication that a linearized analysis of tachyonic fields is inappropriate. Of course, nonlinear field theories with potentials including unstable (or metastable) critical points may play an important role in the paradigm of inflationary cosmology, and such theories can produce interesting asymptotically dS spacetimes [14]. Our point here is simply that one should consider the full nonlinear evolution of such fields including their backreaction on the spacetime geometry.

B. Stability of scalar modes

Here we discuss the stability of dS space with respect to the various scalar field modes introduced in appendix A. In ref. [16], an attempt is made to distinguish the modes associated with $\tilde{\phi}_{0\pm}$ boundary data on the basis of an 'energy functional.' However, one might find this result unsatisfactory given that the 'energy' is not conserved and so it does not give a covariant indicator by which we might measure backreaction effects. So in the following we readdress the question of whether or not these scalar mode functions can successfully be regarded as fluctuations without taking into account any backreaction effects. Our conclusion below is that for the principal or complementary series (i.e., $M^2 > 0$), all of the modes can consistently introduce only a small perturbation throughout the entire time evolution, including the asymptotic dS regions. Therefore, the $\tilde{\phi}_{0\pm}$ modes cannot be distinguished on this basis. Similar arguments were previously sketched out in ref. [51].

To begin, consider a free scalar field propagating in a fixed dS background. In the asymptotic future, i.e., $t \to +\infty$, the background metric (2.7) takes the form

$$ds^2 \simeq -dt^2 + \frac{1}{4}e^{2t/\ell}\, d\Omega_n^2 \,. \tag{B.1}$$

As discussed above, in this asymptotic region, the solutions of scalar field equation (A.2) take the form

$$\phi \simeq e^{-h_\pm t/\ell}\, \tilde{\phi}_{0\pm}(\Omega)\,, \tag{B.2}$$

where as given in eq. (A.8), $h_\pm = n/2 \pm \sqrt{n^2/4 - M^2\ell^2}$. Note that we are assuming that the spatial wavefunctions $\phi_\pm(\Omega)$ have reasonable behavior on the n-sphere, but the precise details will be unimportant.

Consider the contribution of the scalar to the right-hand-side of Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N\, T_{\mu\nu}(\phi) - \Lambda\, g_{\mu\nu}\,. \tag{B.3}$$

Taking ϕ to be a complex scalar field for simplicity, the scalar stress tensor is

$$T_{\mu\nu} = 2\nabla_\mu \phi^* \nabla_\nu \phi - g_{\mu\nu} \left(g^{\sigma\rho}\, \nabla_\sigma \phi^* \nabla_\rho \phi + M^2 |\phi|^2 \right). \tag{B.4}$$

Now to address the stability of the asymptotic dS geometry in eq. (B.1), we will compare the above source in Einstein's equations coming from the scalar field with that arising from the cosmological constant, which defines the background geometry.

In general, one might look at the components of the stress tensor in an orthonormal frame so that they will correspond to the physical energy density and stresses observed by an inertial observer. Note, however, in the metric (B.1) above, the coordinate time t really is the proper time of comoving observers and so by focussing on the energy density T_{tt}, one need not worry about transforming between coordinate and frame indices. Note that the analysis of the pressure components yields the same results as described for the energy density below. Proceeding from eq. (B.4), the relevant energy density is

$$T_{tt} = |\partial_t \phi|^2 + |\hat{\nabla}_k \phi|^2 + M^2 |\phi|^2\,. \tag{B.5}$$

For comparison purposes, we also consider the contribution of the cosmological constant to Einstein's equations (B.3). The corresponding energy density is a constant:

$$T_{tt} = \Lambda = \frac{n(n-1)}{2\ell^2}\,. \tag{B.6}$$

Let us begin by considering scalar mass parameters corresponding to the complementary series, i.e., $0 < M^2\ell^2 < n^2/4$, so that the exponents $h_\pm$ are real. With the wavefunctions given in eq. (B.2), the energy density (B.5) becomes

$$\begin{aligned} T_{tt} &\simeq \frac{1}{\ell^2} \left(h_\pm^2 + M^2\ell^2 \right) e^{-2h_\pm t/\ell}\, |\tilde{\phi}_{0\pm}|^2 + O\left(e^{-2(h_\pm - 1)t/\ell} \right) \\ &\simeq \frac{1}{\ell^2} \left(\frac{n^2}{2} \pm n \sqrt{\frac{n^2}{4} - M^2\ell^2} \right) e^{-2h_\pm t/\ell}\, |\tilde{\phi}_{0\pm}|^2\,. \end{aligned} \tag{B.7}$$

Note that the spatial gradients in the stress tensor (B.5) make subleading contributions above. Now this expression shows that the contribution of the scalar to the local energy density decays in the asymptotically dS region. Certainly this criterion only holds for the complementary series with $0 < M^2\ell^2 < n^2/4$. When the mass is in this range, the perturbation which these modes introduce in the Einstein equations becomes diminishingly small in the asymptotic region. Of course, this result matches the naive intuition that one might derive from the simple observation that the mode functions are decaying as $t \to \infty$. In any event, if the scalar fluctuations begin as small perturbations, they remain a 'small' disturbance throughout the evolution of the spacetime (and the scalar field). Therefore this analysis confirms that the perturbative treatment of the scalar field is consistent and that the asymptotic dS geometry is stable against the introduction of such perturbations.

Similarly, we may consider mass parameters in the range of the principal series, i.e., for $M^2\ell^2 > n^2/4$. Of course, in this case, the exponents are complex: $h_\pm = -n/2 \pm i\omega$. Now, the scalar energy density (B.5) becomes

$$\begin{aligned} T_{tt} &\simeq \left(\frac{|h_\pm|^2}{L^2} + m^2\right) e^{-n\,t/\ell} |\tilde{\phi}_{0\pm}|^2 + \cdots \\ &\simeq 2m^2 e^{-n\,t/\ell}\,|\tilde{\phi}_{0\pm}|^2 \,. \end{aligned} \tag{B.8}$$

Hence we see that in this mass range, in accord with naive expectations, the disturbance of scalar modes to Einstein's equations is small. As above then, we conclude that the asymptotic dS geometry is stable these perturbations.

For a tachyonic field with $M^2 < 0$, the exponents are real but h_- is negative. Therefore the corresponding scalar perturbations are growing exponentially in the asymptotic dS region. The asymptotic scalar energy density is again given by the expression in eq. (B.7). Hence we see that the contribution of the $\tilde{\phi}_{0-}$ modes to the local energy density is growing without bound in this region of the spacetime. Therefore the energy density of the scalar field would quickly overwhelm that of the cosmological constant (B.6) irrespective of how small the perturbations began. Of course, all we can really conclude is that with the growth of these modes, the system will enter a nonlinear regime where the scalar field can no longer be consistently treated using a linearized perturbation analysis. In any event, we interpret this result as indicating the exponential growth of these (or any tachyonic) fields produces an instability as one cannot expect the asymptotic spacetime geometry to resemble dS space (B.1). This confirms the naive expectations for tachyonic fields in dS space. This result is in complete agreement with the previous discussion stating that a successful analysis of such fields must consider the full nonlinear evolution of the scalar including its backreaction on the spacetime geometry.

To reiterate our conclusions, let us note that the analyses here and in appendix A apply to global coordinates on dS space. Further the explicit mode functions for the principal and complementary series given in appendix A are well-behaved and bounded throughout the entire spacetime. Therefore in these cases with suitably 'small' mode coefficients, the above analysis indicates that the scalar energy density (B.5) will be negligible compared to that introduced by the background cosmological constant throughout the spacetime. Hence in these cases, the scalar field will provide a small perturbation throughout the entire evolution of the dS spacetime, not only in the asymptotic regions. It is also clear that this result applies for both the $\tilde{\phi}_{0\pm}$ modes and so these two distinct boundary data cannot be distinguished on this basis.

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